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Solution for 'Topics in complex analysis'

(12/11/2025)

H 9.1 (A general version of Bloch's theorem)
Prove that if $f : U \to \mathbb{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains a ball of radius $(\frac{3}{2} - \sqrt{2})s|f'(c)|$, for every $0 < s < \operatorname{dist}(c, \partial U)$. In particular, show that if $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then $f(\mathbb{C})$ contains balls of arbitrarily large radius.

Solution H 9.1:
Note that the function $g : B_1(0) \to \mathbb{C}$ defined by

$$g(z) = \frac{f(sz + c)}{sf'(c)}$$

is well-defined and can be extended analytically to $B_{1+\eta}(0)$ for $0 < \eta < \frac{\operatorname{dist}(c,\partial U)-s}{s}$. Hence $g \in \mathcal{H}(\overline{B_1(0)})$ and by construction $g'(0) = 1$, so we can apply Bloch's theorem in the standard form, which yields that there exists $p \in \mathbb{C}$ such that $B_{\frac{3}{2}-\sqrt{2}}(p) \subset g(B_1(0))$. It follows that

$$B_{(\frac{3}{2}-\sqrt{2})s|f'(c)|}(sf'(c)p) = sf'(c) \cdot B_{\frac{3}{2}-\sqrt{2}}(p) \subset f(B_s(c)) \subset f(U).$$

This proves the first statement. When f is entire and non-constant, there exists $c \in \mathbb{C}$ such that $f'(c) \neq 0$. Thus we can apply the first statement for every $s > 0$, which proves the second assertion.

□

H 9.2 (Biholomorphic functions from $B_1(0)$ to $B_1(0)$)
For $z_0 \in B_1(0)$ define the function $\varphi_{z_0} : B_1(0) \to \mathbb{C}$, called a Blaschke factor, by

$$\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0}z}.$$

Show that φ_{z_0} is a biholomorphic map from $B_1(0)$ to $B_1(0)$. Moreover, prove that any biholomorphic map $f : B_1(0) \to B_1(0)$ is of the form $f(z) = a\varphi_{z_0}(z)$ for some $z_0 \in B_1(0)$ and $a \in \partial B_1(0)$.

Hint: For the second part, recall the Schwarz lemma (Lemma 7.2 in the notes).

Solution H 9.2:
First note from $|\overline{z_0}z| = |z_0||z| < |z|$ that the denominator never vanishes for $z \in B_1(0)$, so that φ_{z_0} is holomorphic. We next check that φ_{z_0} indeed maps into the unit disc $B_1(0)$ (actually we already did this in Exercise H 7.5). Indeed, we have $|\varphi_{z_0}(z)| < 1$ if and only if

$$|z|^2 - z\overline{z_0} - \overline{z}z_0 + |z_0|^2 < 1 - \overline{z_0}z - z_0\overline{z} + |\overline{z_0}z|^2.$$

Canceling terms we see that $|\varphi_{z_0}(z)| < 1$ is equivalent to

$$|z|^2 + |z_0|^2 < 1 + |z|^2|z_0|^2 \iff 0 < (1 - |z|^2)(1 - |z_0|^2).$$

Since $|z_0| < 1$, this is always true for $z \in B_1(0)$.
To show that φ_{z_0} is bijective we use the remarkable fact that the inverse function is given by φ_{-z_0}, since $\varphi_{z_0} \circ \varphi_{-z_0}(z) = \varphi_{-z_0} \circ \varphi_{z_0}(z) = z$ by direct computation.
In order to prove that any biholomorphic map $f : B_1(0) \to B_1(0)$ is of the form $f(z) = a\varphi_{z_0}(z)$, note that setting $-z_0 = f^{-1}(0)$ it follows that the biholomorphic map $f \circ \varphi_{z_0} : B_1(0) \to B_1(0)$ satisfies

$$f \circ \varphi_{z_0}(0) = f(-z_0) = f \circ f^{-1}(0) = 0.$$

Hence by the Schwarz Lemma it follows that $|f(\varphi_{z_0}(z))| \leq |z|$. Similarly, for the inverse map $\varphi_{z_0}^{-1} \circ f^{-1}$ we also deduce that $|\varphi_{z_0}^{-1}(f^{-1}(y))| \leq |y|$. Inserting $y = f(\varphi_{z_0}(z))$ we obtain that $|f(\varphi_{z_0}(z))| = |z|$, so that again by the Schwarz Lemma there exists a constant $a \in \partial B_1(0)$ such that $f(\varphi_{z_0}(z)) = az$ for all $z \in B_1(0)$. Hence $f(z) = a\varphi_{-z_0}(z)$ for all $z \in B_1(0)$.

□

H 9.3 (Improving the constant in Bloch's theorem)
The purpose of this exercise is to improve the constant $\left(\frac{3}{2} - \sqrt{2}\right)$ appearing in Bloch's theorem. We will show that for every $f \in \mathcal{H}(\overline{B_1(0)})$ with $f'(0) = 1$, the image $f(B_1(0))$ contains a disc of radius $\frac{3}{2}\sqrt{2} - 2$.

a) Motivated by H 9.1, we look for a function $F \in \mathcal{H}(\overline{B_1(0)})$ such that $f(B_1(0)) = F(B_1(0))$ with maximal value $|F'(0)|$. To make this precise, denote by

$$\mathcal{F} := \{f \circ (a\varphi_{z_0}) : z_0 \in B_1(0),\ a \in \partial B_1(0)\},$$

where $\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}$ as in H 9.2. Show that if $h = f \circ (a\varphi_{z_0}) \in \mathcal{F}$ then $h \in \mathcal{H}(\overline{B_1(0)})$, $h(B_1(0)) = f(B_1(0))$, and $|h'(0)| = |f'(-az_0)|(1 - |z_0|^2)$.

b) Denote by q the maximizer of the map $\overline{B_1(0)} \ni z \mapsto |f'(z)|(1 - |z|^2)$. Show that $q \in B_1(0)$ and that denoting $F = f \circ \varphi_{-q}$ it holds that

$$|F'(z)| \leq \frac{|f'(q)|(1 - |q|^2)}{1 - |z|^2}.$$

c) Deduce that $|F'(z)| \leq 2|F'(0)|$ for all $|z| \leq \frac{1}{2}\sqrt{2}$. Conclude the proof using Step 2 of the proof of Bloch's theorem in the lecture notes.

Remark: It is an open problem to find the largest possible radius (say B) allowed in Bloch's theorem. The best known upper and lower bounds are

$$0.4332127... = \frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} \leq B \leq \frac{1}{\sqrt{1 + \sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} = 0.4718617...,$$

and it is conjectured that B is equal to the upper bound.

Solution H 9.3:
a) Each function φ_{z_0} has its only singularity at $\hat{z} = z_0/|z_0|^2$. Note that $|\hat{z}| = |z_0|^{-1} > 1$, so that $\varphi_{z_0} \in \mathcal{H}(\overline{B_1(0)})$. In order to show that the composition $h = f \circ (a\varphi_{z_0})$ belongs to $\mathcal{H}(\overline{B_1(0)})$, we have to ensure that for all $r > 0$ there exists $\eta > 0$ such that $a\varphi_{z_0}(B_{1+\eta}(0)) \subset B_{1+r}(0)$, so that the composition is holomorphic on $B_{1+\eta}(0)$ for $r > 0$ sufficiently small.
Since $|a| = 1$, it suffices to consider $a = 1$. We argue by contradiction, assuming that there exists $r > 0$ such that for all integers $n \geq 1$ there exists $z_n \in B_{1+1/n}(0)$ with $|\varphi_{z_0}(z_n)| \geq 1 + r$. Passing to a subsequence we have that $z_n \to z_\infty \in \overline{B_1(0)}$. We know from Exercise H 9.2 that $\varphi_{z_0}(B_1(0)) = B_1(0)$, so from continuity we deduce that $|\varphi_{z_0}(z)| \leq 1$ for all $z \in \overline{B_1(0)}$. This contradicts the fact that $|\varphi_{z_0}(z_\infty)| \geq 1 + r$, which follows again from continuity. Thus $h \in \mathcal{H}(\overline{B_1(0)})$ as claimed.

The equality $h(B_1(0)) = f(B_1(0))$ follows from the fact that $a\varphi_{z_0} : B_1(0) \to B_1(0)$ is bijective, due to H 9.2 and the fact that multiplication by a (with $|a| = 1$) is a rotation.
Finally, a direct computation yields $\varphi_{z_0}'(0) = 1 - |z_0|^2$, so that by the chain rule

$$|h'(0)| = |f'(a\varphi_{z_0}(0))a\varphi_{z_0}'(0)| = |f'(-az_0)|(1 - |z_0|^2).$$

b) Since the map $z \mapsto |f'(z)|(1 - |z|^2)$ is continuous on $\overline{B_1(0)}$, there exists a maximizer on $\overline{B_1(0)}$. Since this map vanishes on $\partial B_1(0)$ and is positive at $z = 0$, the maximum is achieved at an interior point $q \in B_1(0)$. Setting $F = f \circ \varphi_{-q}$, note that

$$\mathcal{F} = \{F \circ (a\varphi_{z_0}) : \ z_0 \in B_1(0), a \in \partial B_1(0)\}. \tag{1}$$

Indeed, for every $a_1, a_2 \in \partial B_1(0)$ and $z_1, z_2 \in B_1(0)$ the function $(a_1\varphi_{z_1}) \circ (a_2\varphi_{z_2})$ is again a biholomorphic map from $B_1(0)$ to $B_1(0)$ and in H 9.2 we proved that such biholomorphic maps can always be represented in the form $a_3\varphi_{z_3}$ for some $a_3 \in \partial B_1(0)$ and $z_3 \in B_1(0)$.
Thus item a) applies also to F, and implies that for every $z \in B_1(0)$ there exists $z' \in B_1(0)$ and $a \in \partial B_1(0)$ such that

$$\begin{aligned} |F'(z)|(1 - |z|^2) &\overset{a)}{=} |(F \circ (-\varphi_z))'(0)| \\ &\overset{(1)}{=} |(f \circ (a\varphi_{z'}))'(0)| \\ &\overset{a)}{=} |f'(-az')|(1 - |-az'|^2) \\ &\leq |f'(q)|(1 - |q|^2), \end{aligned}$$

where in the last inequality we used the maximality of q. This concludes item b).

c) By item a) we have

$$F'(0) = |f'(q)|(1 - |q|^2),$$

so that b) implies $|F'(z)| \leq 2|F'(0)|$ for all $|z| \leq \frac{1}{\sqrt{2}}$. Since $F'(0) \geq |f'(0)| = 1$, we know that F is non-constant. Hence Step 2 of the proof of Bloch's theorem and a) yield that

$$B_R(F(0)) \subset F(B_{\frac{1}{\sqrt{2}}}(0)) \subset F(B_1(0)) = f(B_1(0))$$

for $R = \frac{3}{2}\sqrt{2} - 2$. This proves the claim.

□

H 9.4 (Optional and difficult)
Let $G \subset \mathbb{C}$ be a simply-connected domain and let $f : G \to \mathbb{C}$ be holomorphic. Set

$$P = \{z \in G : \ f(z) \in \{\pm 1\}\}.$$

Show that there exists a holomorphic function $g : G \to \mathbb{C}$ such that $f = \cos(g)$ if and only if for each $z_0 \in P$ the function $z \mapsto f(z) - f(z_0)$ has a zero of even order at $z = z_0$.

Hint: Show that one can define the function $g(z) = -i \log\left(f(z) + \sqrt{f(z) + 1}\sqrt{f(z) - 1}\right)$. To define the square-root, use the Weierstrass product theorem.

Solution H 9.4:
Assume first that for each $z_0 \in P$ the function $z \mapsto f(z) - f(z_0)$ has a zero of even order. Set $P_+ = \{z \in G : \ f(z) = 1\}$ and $P_- = \{z \in G : \ f(z) = -1\}$. We claim that we can define holomorphic square-roots of $f(z) - 1$ and $f(z) + 1$ on G. Indeed, since P_+ has no accumulation points it follows from the Weierstrass product theorem that there exists a holomorphic function

$h : G \to \mathbb{C}$ such that $Z(h) = P_+$ and the multiplicity of each zero $w \in P_+$ of h coincides with half the multiplicity of the zero of $f(z) - 1$ at $z = w$ (this is still an integer by assumption). Then the function $(f-1)/h^2$ never vanishes on G, so there exists $r : G \to \mathbb{C}$ holomorphic such that

$$f(z) - 1 = e^{r(z)} h(z)^2 = \left(e^{r(z)/2} h(z)\right)^2,$$

where we used that G is simply-connected. By the same argument we can define a holomorphic square root of $f(z) + 1$ on G. Next note that the holomorphic function

$$z \mapsto f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1}$$

never vanishes on G, since otherwise $f(z)^2 = f(z)^2 - 1$, which is impossible. Hence we can define the holomorphic function

$$g(z) = -i \log(f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1})$$

for $z \in G$, as G is simply-connected. Then by the definition of $\cos(w)$ we have

$$\begin{aligned}\cos(g(z)) = \frac{1}{2}\Big[&\exp\Big(-i^2 \log\big(f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1}\big)\Big) \\ &+ \exp\Big(i^2 \log\big(f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1}\big)\Big)\Big] \\ = \frac{1}{2}&\left[f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1} + \frac{1}{f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1}} \right] = f(z).\end{aligned}$$

This proves the first claim.
Now suppose that there exists $z_0 \in P_+$ such that $f(z) - 1$ has a zero of odd multiplicity at $z = z_0$. Then we can write $f(z) - 1 = (z - z_0)^{2k-1} h(z)$ with $h : G \to \mathbb{C}$ holomorphic with $h(z_0) \neq 0$ and $k \in \mathbb{N}$. Assume by contradiction that $f = \cos(g)$ for some holomorphic function $g : G \to \mathbb{C}$. Without loss of generality we may assume that $g(z_0) = 0$ (otherwise just shift $g(z)$ by a multiple of 2π), so denote the multiplicity of this zero by $m \in \mathbb{N}$. Since the power series gives $\cos(z) - 1 = z^2 r(z)$ for some holomorphic function $r : \mathbb{C} \to \mathbb{C}$ with $r(0) \neq 0$, it follows that $\cos(g(z)) - 1$ has a zero at $z = 0$ of order $2m$, which is even. This gives a contradiction.
The case when $f(z)+1$ has a zero of odd multiplicity at $z = z_0 \in P_-$ can be treated in a similar way, since without loss of generality we can assume $g(z_0) - \pi = 0$. Denoting the multiplicity of the zero of $g(z) - \pi$ at $z = z_0$ by $n \in \mathbb{N}$, from $\cos(z) + 1 = (z - \pi)^2 b(z)$ with $b : \mathbb{C} \to \mathbb{C}$ holomorphic and $b(\pi) \neq 0$ we conclude that $\cos(g(z)) + 1$ has a zero at $z = z_0$ of multiplicity $2n$, which is a contradiction.

□

H 9.5 (First applications of Picard's little theorem)
a) Show that every meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that omits three distinct values $a, b, c \in \mathbb{C}$ is constant.

b) Give an example of a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that omits the two values $0, 1 \in \mathbb{C}$.

c) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that $f \circ f$ has a fixed point, except when $f(z) = z + b$ for some $b \in \mathbb{C} \setminus \{0\}$.

Hint: Consider the function $z \mapsto \frac{f(f(z))-z}{f(z)-z}$. Show that it is equal to a constant c and differentiate it. Then deduce that $f' \circ f$ omits 0 and c. Finally, show that this implies that f' is constant.

Solution H 9.5:
a) Assume that f is non-constant. Note that the function $z \mapsto (f(z) - a)^{-1}$ can be extended

to a non-constant entire function which omits the two distinct values $(b-a)^{-1}$ and $(c-a)^{-1}$. This contradicts Picard's little theorem.

b) Consider for instance the meromorphic function $f(z) = (1+e^z)^{-1}$, which has first order poles at $z_n = (2n+1)\pi i$ for $n \in \mathbb{Z}$.

c) Suppose that $f \circ f$ has no fixed points. Then the function

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$

is entire and $g(z) \neq 0$ for all $z \in \mathbb{C}$. Moreover, note that $g(z) \neq 1$ for all $z \in \mathbb{C}$. Indeed, $g(z) = 1$ implies that $f(z)$ is a fixed point for f and therefore also for $f \circ f$. Hence by Picard's little theorem $g(z) = c \in \mathbb{C} \setminus \{0, 1\}$. Differentiation yields

$$f'(z)\big(f'(f(z)) - c\big) = 1 - c.$$

Since $c \neq 1$ we know that $f'(z) \neq 0$ for all $z \in \mathbb{C}$ and $f'(f(z)) \neq c$ for all $z \in \mathbb{C}$. Thus $f' \circ f$ is entire and omits the values 0 and $c \neq 0$. Picard's little theorem implies that it is constant. Combining the open mapping theorem (for f) and the identity theorem (for f'), it follows that f' is constant. Hence $f(z) = az + b$ for all $z \in \mathbb{C}$. Since f has no fixed point, we conclude that $a = 1$ and $b \neq 0$ as claimed. $\square$